

# TURAEV-VIRO INVARIANTS AS AN EXTENDED TQFT II

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ABSTRACT. In this paper, we present the next step in the proof that  $Z_{TV,C} = Z_{RT,Z(C)}$ , namely that the theories give the same 3-manifold invariants. In future papers we will show that this equality extends to an equivalence of TQFTs. This paper is a continuation of [BK].

## INTRODUCTION

In [BK], we defined a version of the Turaev-Viro TQFT for 3-manifolds with corners. The theory coincides with the classical TV for ordinary 3-manifolds with boundary, and thus takes as input data a spherical category  $\mathcal{C}$ . Instead of considering manifolds with corners, we consider ordinary 3-manifolds with boundary, replacing the corners with framed embedded tubes. As in [BK2001], these are the same extended 3-manifolds used in the Reshetikhin-Turaev (RT) theory.

We also made use of a category associated to  $\mathcal{C}$ , its Drinfeld Center  $Z(\mathcal{C})$ . If  $\mathcal{C}$  is spherical, it can be shown that  $Z(\mathcal{C})$  is modular, and, in particular, is braided. The extended theory assigns the category  $Z(\mathcal{C})$  to a circle, and we computed that for the  $n$ -punctured sphere with boundary components labeled  $Y_1, \dots, Y_n$ ,  $Z_{TV,C}(S_n^2, Y_1, \dots, Y_n) = \text{Hom}_{Z(\mathcal{C})}(\mathbf{1}, Y_1, \dots, Y_n)$ , the same space that  $Z_{RT,Z(\mathcal{C})}$  produces.

In this paper, we finish proving that for a closed 3-manifold  $\mathcal{M}$  (possibly with an embedded link inside),  $Z_{TV,C}(\mathcal{M}) = Z_{RT,Z(\mathcal{C})}(\mathcal{M})$ . Turaev and Virelizier recently posted a proof of this formula [TV], but the methods are different, and in particular involves state sums on skeletons of 3-manifolds and the theory of Hopf monads in monoidal category. In contrast, this paper employs the methods from [BK], in which the Turaev-Viro TQFT is described as a 3-2-1 extended theory in the sense of [Lur]. This extended theory is a generalization of the classical results of [TV1992] and [BW1996].

The organization of this paper is as follows. In section 1, we recall some definitions and results that will be used throughout the paper. In section 2, we prove that the theories coincide for  $S^3$  with a link inside. We do this by decomposing  $S^3$  into a finite collection of *building blocks*, demonstrating that the theories coincide on these blocks, and then using the gluing axiom. In section 3, we describe graphically the vector space assigned to the torus and examine the action of the mapping class group. In particular, we show that the generators  $T$  and  $S$  of the mapping class group act by multiplication by the twist and  $s$ -matrices respectively, just as they do in  $Z_{RT}$ . Finally, in the last section, we prove the surgery formula for our theory, which implies the main theorem as a corollary. The appendix contains some of the more detailed computations.

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## 1. PRELIMINARIES

In this section, we recall without proof some important results that will be useful in this paper. All proofs may be found in [BK], which is a prerequisite to reading this paper.

Throughout the paper  $\mathcal{C}$  will denote a spherical fusion category over some algebraically closed field of characteristic 0. Recall that this means that  $\mathcal{C}$  is semisimple, with finitely many isomorphism classes of simple objects. We will also assume that  $\mathcal{C}$  is strict pivotal and that the unit object  $\mathbf{1} \in \mathcal{C}$  is simple. For  $X \in \mathcal{C}$ , we denote by

$$d_X = \dim X = \text{tr}(Id_X) \in \mathbf{k}$$

For each simple object  $X_i \in \mathcal{C}$ , we fix a choice of square root  $\sqrt{d_{X_i}}$  so that that for  $\mathbf{1} \in \mathcal{C}$ ,  $\sqrt{d_{\mathbf{1}}} = 1$ , and for any simple object  $X$ ,  $\sqrt{d_X} = \sqrt{d_{X^*}}$ . We also fix an element

$$(1.1) \quad \mathcal{D} = \sqrt{\sum_{x \in \text{Irr}(\mathcal{C})} d_x^2}$$

which we call the dimension of  $\mathcal{C}$  or  $\text{Dim}(\mathcal{C})$ . By results of [ENO2005],  $\mathcal{D} \neq 0$  and for simple  $X$ ,  $d_X \neq 0$ . For a category  $\mathcal{C}$ , we define a functor  $\mathcal{C}^{\boxtimes n} \rightarrow \text{Vec}$  by

$$(1.2) \quad \langle V_1, \dots, V_n \rangle_{\mathcal{C}} = \text{Hom}_{\mathcal{C}}(\mathbf{1}, V_1 \otimes \dots \otimes V_n)$$

where  $V_1, \dots, V_n \in \mathcal{C}$ . The subscript  $\mathcal{C}$  is included in 1.2 to remind the reader which category we are operating in. We will omit it when there is no potential ambiguity. For any objects  $A, B \in \text{Obj } \mathcal{C}$ , we have a non-degenerate pairing  $\text{Hom}_{\mathcal{C}}(A, B) \otimes \text{Hom}_{\mathcal{C}}(A^*, B^*) \rightarrow \mathbf{k}$  given by

$$(1.3) \quad (\varphi, \varphi') = (\mathbf{1} \xrightarrow{\text{coev}_A} A \otimes A^* \xrightarrow{\varphi \otimes \varphi'} B \otimes B^* \xrightarrow{\text{ev}_B} \mathbf{1})$$

**Lemma 1.1.** *Let  $X$  be a simple object. Define the composition map*

$$(1.4) \quad \begin{aligned} \langle V_1, \dots, V_n, X \rangle \otimes \langle X^*, W_1, \dots, W_m \rangle &\rightarrow \langle V_1, \dots, V_n, W_1, \dots, W_m \rangle \\ \varphi \otimes \psi &\mapsto \varphi \bullet_X \psi = \sqrt{d_X} \text{ev}_X \circ (\varphi \otimes \psi) \end{aligned}$$

*Then the composition map agrees with the pairing:*

$$(\varphi \bullet_X \psi, \psi' \bullet_{X^*} \varphi') = (\varphi, \varphi')(\psi', \psi)$$

In addition to the category  $\mathcal{C}$ , our extended TV theory makes use of a related category, the *Drinfeld Center* of  $\mathcal{C}$ . Let  $\mathcal{C}$  be a spherical fusion category. The *Drinfeld Center* of  $\mathcal{C}$ , denoted  $Z(\mathcal{C})$  is the category with

- Objects are pairs  $(Z, \varphi_Z)$ , where  $Z \in \text{Obj } \mathcal{C}$ , and  $\varphi_Z : Z \otimes X \rightarrow X \otimes Z$  a natural isomorphism for all  $X \in \mathcal{C}$ . The collection of morphisms  $\{\varphi_Z\}_{Z \in Z(\mathcal{C})}$  is aptly named a half-braiding, and these morphisms must satisfy certain coherence conditions.
- Morphisms  $\Psi : (Z, \varphi_Z) \rightarrow (Y, \varphi_Y)$  in  $Z(\mathcal{C})$  are morphisms  $\Psi : Z \rightarrow Y$  in  $\mathcal{C}$  that are compatible with the half-braiding.

For a more precise definition see [Müg2003b]. The following important theorem is due to Mueger.

**Theorem 1.2.**  *$Z(\mathcal{C})$  is a modular category; in particular, it is semisimple with finitely many simple objects, it is braided and has a pivotal structure which coincides with the pivotal structure on  $\mathcal{C}$ .*

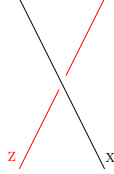


FIGURE 1. The half-braiding  $\varphi_Z : Z \otimes X \rightarrow X \otimes Z$

We will make heavy use of the graphical techniques developed in, e.g. [Tur1994] to represent morphisms in the category  $\mathcal{C}$ . These techniques will greatly simplify the calculations that follow. As in [BK], morphisms in the category of tangles are read bottom to top. We also allow for circular coupons in addition to the standard rectangular ones. Tangle strands are to be labelled by objects of  $\mathcal{C}$  and coupons, with morphisms in the appropriate  $Hom$  spaces (which are vector spaces in a spherical category). Note that  $\mathcal{C}$  is not equipped with a braiding, so we will not allow edges to cross. When evaluating such morphisms, we use the following conventions:

- (1) If a figure contains a pair of circular coupons, one with outgoing edges labelled  $V_1, \dots, V_n$  and the other with edges labelled  $V_n^*, \dots, V_1^*$  and the coupons are labelled by a pair of letters, such as  $\varphi$  and  $\varphi^*$ , it stands for summation over dual bases with respect to 1.3. Alternatively, for sake of brevity we will on occasion represent such pairs of circular coupons by pairs of vertices of the same color.
- (2) For a spherical fusion category  $\mathcal{C}$ , we refer to the set of isomorphism classes of simple objects as  $\text{Irr}(\mathcal{C})$ . By an abuse of notation, we use the term *simple object* to denote an element of this set
- (3) If a diagram contains an unlabelled edge, we sum over all possible labellings of that edge with simple objects  $X_i \in \mathcal{C}$ , each with weight  $d_i$ .
- (4) When labelling edges with simple object  $X_i$  we will often just use the label  $i$ .
- (5) We will sometimes neglect to orient edges in diagrams when doing so would prove cumbersome. The orientations are only important when, for example, using the composition map or pairing dual vertices, and we will be careful in these cases.

The following lemma will be very useful in forthcoming computations.

**Lemma 1.3.**

- (1) *If  $X$  is simple and  $\varphi \in \langle X, A \rangle$ ,  $\varphi' \in \langle A^*, X^* \rangle$  then*

$$\begin{array}{c}
 \begin{array}{c} X \quad A \quad A^* \quad X^* \\ \curvearrowright \\ \begin{array}{|c|} \hline \varphi \end{array} \quad \begin{array}{|c|} \hline \varphi' \end{array} \end{array} \\
 \end{array} = \frac{(\varphi, \varphi')}{d_X} \begin{array}{c} X \quad X^* \\ \curvearrowright \end{array}$$

(2)

$$\sum_{i \in \text{Irr}(\mathcal{C})} d_i \begin{array}{c} V_1 \quad V_n \\ \swarrow \quad \searrow \\ \textcircled{\varphi} \\ \xrightarrow{X_i} \textcircled{\varphi^*} \\ \nwarrow \quad \nearrow \\ V_n \quad V_1 \end{array} = \begin{array}{c} \cdots \quad \cdots \\ \nearrow \quad \nwarrow \\ V_n \\ \nwarrow \quad \nearrow \\ V_1 \end{array}$$

(3) If the subgraphs  $A, B$  are not connected, then

$$\begin{array}{c} A \quad V_1 \\ \text{---} \textcircled{\varphi} \\ \text{---} V_n \end{array} \quad \begin{array}{c} V_1 \\ \nearrow \quad \nwarrow \\ \textcircled{\varphi^*} \\ \nwarrow \quad \nearrow \\ V_n \quad B \end{array} = \begin{array}{c} A \quad V_1 \quad B \\ \text{---} \textcircled{\varphi} \quad \textcircled{\varphi^*} \text{---} \\ \text{---} V_n \end{array}$$

We also have the following isomorphism which can be realized graphically.

**Lemma 1.4.** For any  $A, B \in \text{Obj } \mathcal{C}$ , the map

$$\bigoplus_{Z \in \text{Irr}(Z(\mathcal{C}))} \langle Z, A \rangle \otimes \langle Z^*, B \rangle \rightarrow \bigoplus_{X \in \text{Irr}(\mathcal{C})} \langle A, X, B, X^* \rangle$$

(1.5)

$$\varphi \otimes \psi \mapsto \bigoplus_{X \in \text{Irr}(\mathcal{C})} \frac{\sqrt{d_X} \sqrt{d_Z}}{\mathcal{D}} \begin{array}{c} A \quad X \quad B \\ | \quad \uparrow \quad | \\ \textcircled{\varphi} \text{---} Z \text{---} \textcircled{\psi} \\ \text{---} \end{array}$$

is an isomorphism.

As shown in [RT], we one can *evaluate* a graph  $(\Gamma; X_i)$  with edges colored by irreducible objects of  $\mathcal{C}$  and with vertices (or "coupons") labeled by appropriate morphisms to obtain a number  $F((\Gamma; X_i)) \in \mathbf{k}$ . The axioms of a spherical category imply that we may actually view the graph as embedded in the sphere; any two *flattening*s to a planar graph will evaluate to the same number. In [BK] the authors considered a generalization of these graphs.

**Definition 1.5.** An *extended graph*  $\hat{\Gamma}$  consists of a usual graph  $\Gamma$  embedded in  $S^2$  along with a finite collection of strands  $\{\gamma\}$  that terminate on vertices of  $\Gamma$ . These strands are allowed to intersect the edges of  $\Gamma$  away from vertices, and may cross over or under one another.

In this paper, the strands  $\{\gamma\}$  will be colored red. A labelling of an extended graph  $\hat{\Gamma}$  is function  $\eta$  which assigns to each oriented edge of  $\Gamma$  an object of  $\mathcal{C}$ , to each red strand, an object of  $Z(\mathcal{C})$ , and to each vertex or coupon, a morphism in the appropriate Hom space (all Homs in  $\mathcal{C}!$ ). Given a labelled extended graph  $(\hat{\Gamma}, \eta)$  we may flatten the graph by removing a point in  $S^2$  and evaluate the graph as in [RT], where we replace all crossings with the half-braiding.

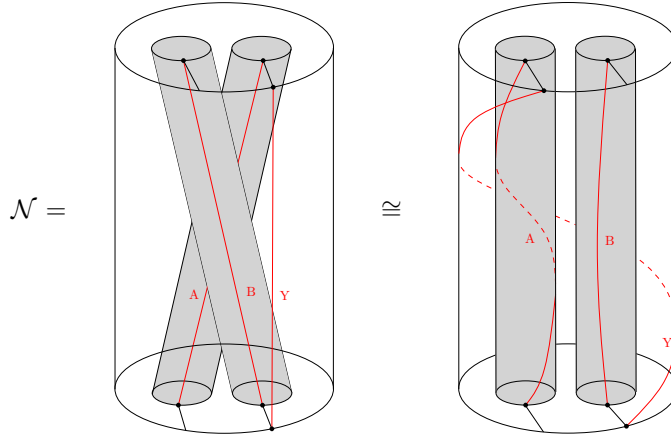
**Theorem 1.6.** The number  $Z_{RT}(\hat{\Gamma}) \in \mathbf{k}$  does not depend on the choice of a point to remove from  $S^2$  and thus defines an invariant of colored extended graphs on the sphere. Moreover, this number is invariant under homotopy of strands  $\{\gamma\}$ .



## 2. SPHERE

In this section we compute the TV state sum for several extended 3-manifolds, which we call *generators*. If  $S_L^3$  denotes  $S^3$  with an embedded link  $L$  inside, we can decompose  $S_L^3$  into a finite union of these generators. Using the computations in this section and the gluing axiom for our TQFT, we conclude that the theories give the same answer for  $S_L^3$ . In what follows, all links are framed and oriented.

Consider the following extended 3-manifold structure on  $\mathcal{N}$  where  $\mathcal{N}$  is the cobordism between 3-punctured spheres that interchanges two of the embedded disks with longitudes labeled as pictured <sup>1</sup>. Clearly, the two picture are homeomorphic. If we take a modular category as input data, Reshetikhin-Turaev theory



gives  $Z_{RT}(\mathcal{N}) = Id_Y \otimes \sigma_{AB}$ , where  $\sigma$  is the braiding [BK2001]. We now show that Turaev-Viro theory gives the same answer.

**Lemma 2.1.** *Let  $\mathcal{C}$  be a spherical category. Then there is a canonical isomorphism  $Z_{TV,\mathcal{C}}(\mathcal{N}) \cong Z_{RT,Z(\mathcal{C})}(\mathcal{N})$*

*Proof.* See Section 5 □

Now let  $\mathcal{M} = S^2 \times I$  with a single open embedded tube colored by  $Y \in Z(\mathcal{C})$  as shown. As in the previous lemma, we have removed a solid cylinder from  $\mathcal{M}$ . By definition,  $Z_{RT,Z(\mathcal{C})}(\mathcal{M}) = ev_Y : Y^* \otimes Y \rightarrow \mathbf{1}$ , the evaluation map.

**Lemma 2.2.**  $Z_{TV,\mathcal{C}}(\mathcal{M}) \cong Z_{RT,Z(\mathcal{C})}(\mathcal{M})$ .

*Proof.* See Section 5 □

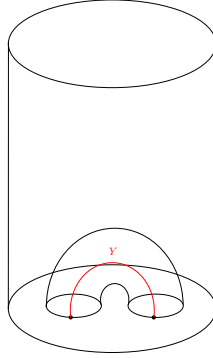
It follows by an identical calculation that  $Z_{TV}(\mathcal{M}')$  gives the coevaluation map, where  $\mathcal{M}'$  is similar to  $\mathcal{M}$  but inverted.

We can now use the above two lemmas to state the following result.

**Theorem 2.3.** *Let  $M = S_L^3$  be the 3-sphere with an embedded link  $L$  inside with components colored by  $Y \in Irr(Z(\mathcal{C}))$ . Then  $Z_{TV,\mathcal{C}}(M) = Z_{RT,Z(\mathcal{C})}(M)$ .*

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<sup>1</sup>We have removed a solid cylinder from the figure to make the diagram more manageable



*Proof.* After isotoping  $L$  appropriately, we can cut  $M$  into regions, each of which is isomorphic to  $\mathcal{N}$ ,  $\mathcal{M}$  or  $\mathcal{M}'$  from the above lemmas or to  $B^3$ . The theorem then follows from the lemmas and the gluing axiom [BK].  $\square$

Note that it is known that for RT,  $Z_{RT, \mathcal{A}}(S_L^3) = \frac{1}{\text{Dim}(\mathcal{A})} F(L)$ , so the theorem gives

$$(2.1) \quad Z_{TV, \mathcal{C}}(S_L^3) = Z_{RT, Z(\mathcal{C})}(S_L^3) = \frac{1}{\text{Dim}(Z(\mathcal{C}))} F(L) = \frac{1}{\mathcal{D}^2} F(L)$$

which can also be verified by direct computation. Here we use the fact that  $\text{Dim}(Z(\mathcal{C})) = \text{Dim}(\mathcal{C})^2$  [Müg2003b].

### 3. COMPUTATIONS WITH THE TORUS

In the previous section, we established that  $Z_{RT}(S_L^3) = Z_{TV}(S_L^3)$ , where  $S_L^3$  is the 3-sphere with an embedded link  $L$  inside. It is a classical result that any connected, closed 3-manifold may be obtained from  $S^3$  via surgery along a framed link  $L$ , or, more precisely, along a tubular neighborhood of  $L$ . A tubular neighborhood of a link is simply a disjoint union of solid tori. In this section, we study the vector space  $Z_{TV}(\mathbb{T}^2)$  and describe graphically an inner product, an orthonormal basis and action of the mapping class group (MCG) of the torus. We denote the standard 2-torus  $S^1 \times S^1$  by  $\mathbb{T}^2$ .

The following result is well-known (see e.g. [Müg2003b]):

**Lemma 3.1.**  $Z_{TV, \mathcal{C}}(\mathbb{T}^2)$  has basis indexed by isomorphism classes of irreducible objects of  $Z(\mathcal{C})$ .

*Proof.* The torus may be obtained from the 2-punctured sphere,  $S_2^2$  by gluing together its two boundary circles. It follows directly from the gluing axiom for surfaces (Theorems 8.4, 8.5 in [BK]) that

$$(3.1) \quad Z_{TV, \mathcal{C}}(\mathbb{T}^2) = \bigoplus_{Z \in \text{Irr}(Z(\mathcal{C}))} \langle Z, Z^* \rangle_{Z(\mathcal{C})}.$$

$\square$

We can also see this explicitly by computing  $Z_{TV}(\mathbb{T}^2 \times I)$ , as is done in Section 5.

We denote by  $\mathbf{T}_Z^2$  the solid torus with a closed embedded tube inside with (untwisted) longitude labeled by  $Z \in \text{Irr}(Z(\mathcal{C}))$  as shown in Figure 2. Then  $Z_{TV, \mathcal{C}}(\mathbf{T}_Z^2)$  is a vector in  $Z_{TV, \mathcal{C}}(\mathbb{T}^2)$ . We denote this vector by  $[Z]$ .

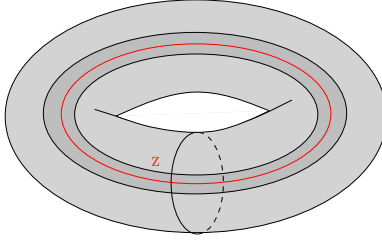


FIGURE 2. The solid torus with a closed embedded tube labelled by  $Z \in Z(\mathcal{C})$

**Lemma 3.2.**  $\{[Z]\}_{Z \in \text{Irr}(Z(\mathcal{C}))}$  form a basis in  $Z(\mathbb{T}^2)$ .

A simple computation shows that  $[Z] = \frac{1}{\sqrt{d_Z}} \text{coev}_Z$ , under the identification 3.1. Note that  $\text{coev}_{\mathcal{C}}|_{Z(\mathcal{C})} = \text{coev}_Z$ .

#### 4. SURGERY

If  $M, N$  are manifolds with boundary and  $\varphi : \partial M \rightarrow \partial N$  is a homeomorphism, we may glue  $M$  and  $N$  along their boundaries to obtain a closed 3-manifold, denoted  $M \sqcup_{\varphi} N$ . Such a map  $\varphi$  induces a linear map between the corresponding vector spaces, and we denote this map  $\varphi_* : Z(\partial M) \rightarrow Z(\partial N)$ . By the gluing axiom,  $Z(M \sqcup_{\varphi} N) = (\varphi_* Z(M), Z(N))$ .

**Lemma 4.1.**  $M \sqcup_{\varphi} N$  only depends on the isotopy class of  $\varphi$

For a proof of this lemma, see [BK2001].

**Definition 4.2.** For a closed manifold  $M$ , the *mapping class group*  $\Gamma(M)$  is the group of isotopy classes of homeomorphisms  $M \rightarrow M$ .

Since any 3-manifold may be obtained from  $S^3$  by surgery along an embedded link, or equivalently along a collection of solid tori, we will be primarily concerned with the following example:

**Example 4.3.**  $\Gamma(\mathbb{T}^2) = SL_2(\mathbb{Z})$ , which is generated by  $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$  and  $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ . If we pick generators  $\alpha, \beta$  for  $H_1(\partial \mathbb{T}^2, \mathbb{Z})$  as in Figure 3, then  $S$  acts by interchanging<sup>2</sup>  $\alpha$  and  $\beta$  and  $T$  is a Dehn twist (See Figure 3). Further, the action of  $T$  extends to a homeomorphism of the solid torus.

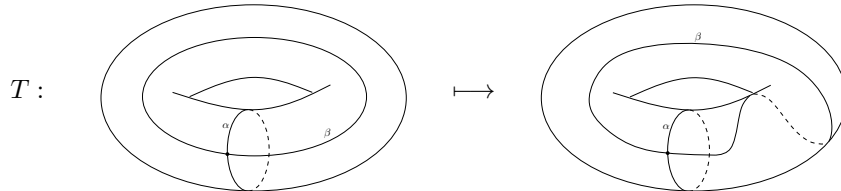


FIGURE 3. The action of  $T$  on the Torus

<sup>2</sup>More precisely  $S(\alpha) = \beta, S(\beta) = -\alpha$



We now describe an inner product on  $Z(\mathbb{T}^2)$ .

**Lemma 4.4.** *Define a bilinear form on  $Z(\mathbb{T}^2)$  by  $\langle [Z], [W] \rangle = Z(\mathbf{T}_Z^2 \sqcup_U \mathbf{T}_W^2)$ , where  $U = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ . Then  $\{[Z]\}_{Z \in \text{Irr}(Z(\mathcal{C}))}$  is orthonormal with respect to this form.*

*Proof.* See Section 5. □

Since  $Z(\mathcal{C})$  is a modular category, it has an s-matrix  $\tilde{s}$  ([BK2001]) and a twist matrix  $t$  where  $t_{i,j}$  is zero for  $i \neq j$  and is defined for  $i = j$  by

$$\text{Diagram} = t_{i,i}$$

Let  $S$  and  $T$  be as above. The following theorem shows that  $S$  and  $T$  act on  $Z(\mathbb{T}^2)$  by the s-matrix and the twist matrix respectively.

**Theorem 4.5.** (1)  $S_*[Z] = \sum_{W \in \text{Irr}(Z(\mathcal{C}))} \frac{\tilde{s}_{ZW}}{\mathcal{D}^2} [W]$ .  
 (2)  $T_*[Z] = t_{Z,Z}[Z]$

*Proof.* (1) By the gluing axiom,  $(S_*[Z], [W]) = Z_{TV,C}(\mathbf{T}_Z^2 \sqcup_S \mathbf{T}_W^2) = Z_{TV,C}(S_L^3)$ , where  $L$  is the Hopf link with components labelled by  $Z$  and  $W$ . By 2.3, this equals  $Z_{RT,Z(\mathcal{C})}(S_L^3) = \tilde{s}_{ZW}$ .  
 (2) By the gluing axiom  $(T_*[Z], [W]) = Z_{TV,C}(\mathbf{T}_Z^2 \sqcup_T \mathbf{T}_W^2)$ . This manifold is homeomorphic to  $S^2 \times S^1$  with two unlinked embedded closed tubes labelled  $Z$  and  $W$  respectively, the  $Z$  tube with a single positive twist. This follows directly from the fact that  $\mathbf{T}^2 \sqcup_U \mathbf{T}^2 = S^2 \times S^1$  combined with the fact that the map  $T$  extends to a homeomorphism of solid tori. Replacing the twist with a multiplicative factor  $t_{Z,Z}$ , we get  $Z_{TV,C}(\mathbf{T}_Z^2 \sqcup_T \mathbf{T}_W^2) = t_{Z,Z} Z_{TV,C}(\mathbf{T}_Z^2 \sqcup_U \mathbf{T}_W^2) = t_{Z,Z} \delta_{Z,W}$ , where the final equality holds by Lemma 4.4. □

The following computations will be particularly useful:

- (1)  $S_*[\mathbf{1}] = \sum_{W \in \text{Irr}(Z(\mathcal{C}))} \frac{d_W}{\mathcal{D}^2} [W]$ . This follows immediately from Theorem 4.5(1) and the fact that  $\tilde{s}_{\mathbf{1},Z} = d_Z$ .  
 (2)  $S_* \sum_Z \frac{d_Z}{\mathcal{D}^2} [Z] = [\mathbf{1}]$ . This follows from the equation  $\sum_Z d_Z \tilde{s}_{Z,W} = \delta_{W,\mathbf{1}} \mathcal{D}^2$  (See [BK2001], Chapter 3).

Let us briefly outline what we have accomplished so far.

- (1) In Section 2, we demonstrated that  $Z_{TV,C}(S_L^3) = Z_{RT,Z(\mathcal{C})}(S_L^3)$ . This was done by decomposing  $S_L^3$  as a union of several *types* of 3-manifolds with boundary, showing the result holds for each type, and using the gluing axiom.

- (2) In Section 3 we showed that  $Z_{TV,C}(\mathbb{T}^2) \cong Z_{RT,Z(C)}(\mathbb{T}^2)$  and that this isomorphism agrees with the action of the mapping class group  $\Gamma(\mathbb{T}^2)$

We will now connect these results. Let  $K$  be a framed knot with framing  $n \in \mathbb{Z}$  and  $\mathcal{M}_K$  a 3-manifold with  $K$  embedded. Let  $\mathcal{M}'$  be the result of performing surgery in  $\mathcal{M}$  along  $K$ . We can write the gluing map  $\varphi : \mathbb{T}^2 \rightarrow \partial(\mathcal{M} - \mathbb{T}^2)$  as  $\varphi = T^n \circ S$ .

**Lemma 4.6.**  $Z_{TV,C}(\mathcal{M}') = \frac{1}{\mathcal{D}^2} Z_{TV,C}(\mathcal{M}_K)$ .

*Proof.* Recall that  $Z_{TV,C}(\mathcal{M}_K) \equiv \sum_i d_i Z_{TV,C}(\mathcal{M}_{K_i})$ , where  $K_i$  is the knot  $K$  colored by  $i \in \text{Irr}(\mathcal{C})$  and  $\mathcal{M}_K = (\mathcal{M} - \mathbf{T}^2) \sqcup_{Id} \mathbf{T}^2$ , where  $\mathbf{T}^2$  is a tubular neighborhood of  $K$ . Then

$$\begin{aligned} Z_{TV,C}(\mathcal{M}_K) &= \left( \sum d_i [i], Z(\mathcal{M}_K - \mathbf{T}^2) \right) = \left( \sum d_i t_{i,i}^n [i], Z(\mathcal{M}_K - \mathbf{T}^2) \right) \\ &= \mathcal{D}^2 (T_*^n S_*[1], Z(\mathcal{M}_K - \mathbf{T}^2)) = \mathcal{D}^2 Z_{TV,C}((\mathcal{M} - \mathbf{T}^2) \sqcup_{T^n \circ S} \mathbf{T}^2) = \mathcal{D}^2 Z_{TV,C}(\mathcal{M}'). \end{aligned}$$

□

We can slightly generalize Lemma 4.6. The proof is similar.

**Lemma 4.7.** *Let  $\mathcal{M}_L$  be a 3-manifold with a framed link  $L$  inside. Let  $\mathcal{M}'_{L'}$  be the result of performing surgery on  $\mathcal{M}_L$  along a single component of  $L$ . (Note that  $|L'| = |L| - 1$ ). Then  $Z_{TV,C}(\mathcal{M}'_{L'}) = \frac{1}{\mathcal{D}^2} Z_{TV,C}(\mathcal{M}_L)$ .*

Note that if  $L$  is a knot, the lemma reduces to Lemma 4.6. Now we state the main and final theorem of the paper, which relates the Reshetikhin-Turaev and Turaev-Viro invariants. The proof makes repeated use of Lemma 4.7 and is very simple. Since we will be working with two categories,  $\mathcal{C}$  and its Drinfeld Center  $Z(\mathcal{C})$ , we will replace all potentially ambiguous shorthand in what follows. For example, we will write  $\text{Dim}(\mathcal{C})$  instead of  $\mathcal{D}$ .

**Theorem 4.8.** *Let  $\mathcal{M}$  be a closed, oriented 3-manifold with a colored link inside. Then  $Z_{TV,C}(\mathcal{M}) = Z_{RT,Z(C)}(\mathcal{M})$ .*

*Proof.* For simplicity, we can assume  $\mathcal{M}$  has no embedded link. The general case is proved in the exact same fashion. We can obtain  $\mathcal{M}$  from  $S_L^3$  via surgery along a tubular neighborhood of  $L$ . Evidently, we can perform surgery along each component of  $L$  individually. Using Lemma 4.7 repeatedly, we obtain

$$Z_{TV,C}(\mathcal{M}) = \frac{1}{\text{Dim}(\mathcal{C})^{2|L|}} Z_{TV,C}(S_L^3) = \sum \frac{1}{\text{Dim}(\mathcal{C})^{2|L|}} \prod d_{Y_i} Z_{TV,C}(S^3; L_i; Y_i).$$

By Theorem 2.3, this equals

$$\begin{aligned} \sum \frac{1}{\text{Dim}(\mathcal{C})^{(2|L|+2)}} \prod d_{Y_i} F(L_i; Y_i) &= \sum \frac{1}{\text{Dim}(Z(\mathcal{C}))^{(|L|+1)}} \prod d_{Y_i} F(L_i; Y_i) \\ &\equiv Z_{RT,Z(C)}(\mathcal{M}). \end{aligned}$$

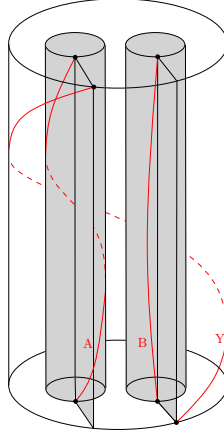
□

We have shown that  $Z_{TV,C}$  and  $Z_{RT,Z(C)}$  give the same 3-manifold invariants. Using the gluing axiom and the fact that the theories agree on the  $n$ -punctured sphere, one can easily show that for  $\Sigma_g$  a closed genus  $g$  surface,  $\text{Dim}(Z_{TV,C}(\Sigma_g)) =$

$\text{Dim}(Z_{RT,Z(C)}(\Sigma_g))$ <sup>3</sup>. Thus, the vector spaces associated to 2-manifolds are isomorphic. This is not enough, however, to show the TQFTs are isomorphic. We will have to construct a canonical isomorphism between the spaces and show these satisfy certain compatibility conditions. This will be done in an upcoming paper [Bal].

## 5. APPENDIX: SOME PROOFS

In this appendix, we include some of the longer computations described in the paper.



**Proof of Lemma 2.1.** Decompose  $\mathcal{N}$  into a combinatorial 3-manifold. The tubes labeled  $A$  and  $B$  are shaded gray and the tube labeled  $Y$  lies on the outside of the 3-cell. The decomposition has 4 vertices and 16 edges, of which 4 are internal and 12 lie on the boundary. The decomposition has four 3-cells, of which 3 are labeled tubes. Orienting and coloring edges, we get 4 graphs, one for each 3-cell. Note that in the following diagrams we often replace coupons with vertices and omit labeling vertices by the appropriate vectors. As always, we label vertices corresponding to dual Hom-spaces with dual basis vectors and sum over all these 'paired' bases (Recall that these correspond to internal 2-cells). We also label the bottom and top vertices by  $\varphi$  and  $\varphi'$  respectively.

The state sum is therefore:

$$Z_{TV}(\mathcal{N}) = \sum D K_3 K_Y K_A K_B$$

where  $K_3, K_Y, K_A, K_B$  denote the evaluations of each of the labeled graphs picture above and  $D$  is the following unsightly term:

$$D = \mathcal{D}^{-8} \sum d_{x_1} d_{z_1} d_{\alpha_3} d_{\beta_3} d_{\alpha_1}^{\frac{1}{2}} d_{\alpha_2}^{\frac{1}{2}} d_{\beta_1}^{\frac{1}{2}} d_{\beta_2}^{\frac{1}{2}} d_{x_2}^{\frac{1}{2}} d_{x_3}^{\frac{1}{2}} d_{z_2}^{\frac{1}{2}} d_{z_3}^{\frac{1}{2}} d_{u_1}^{\frac{1}{2}} d_{u_2}^{\frac{1}{2}} d_{v_1}^{\frac{1}{2}} d_{v_2}^{\frac{1}{2}}$$

Recall that each vertex is labeled by a morphism in the corresponding Hom-space. Pairing dual morphisms, we can glue in the 3 tubes.

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<sup>3</sup>In fact it is not hard to explicitly compute this this common dimension. See [Tur1994]

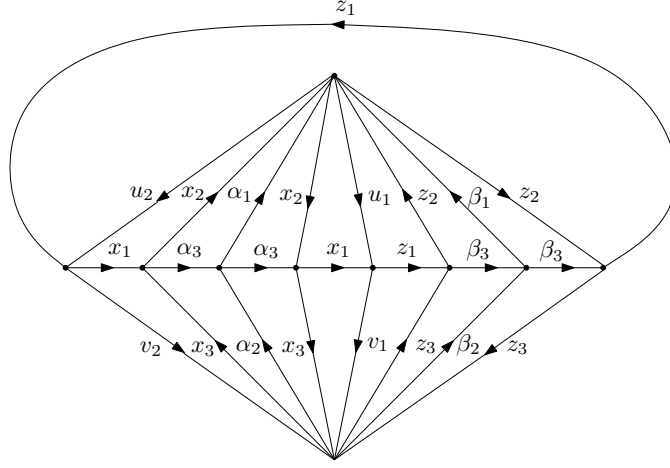


FIGURE 4. Big 3-cell

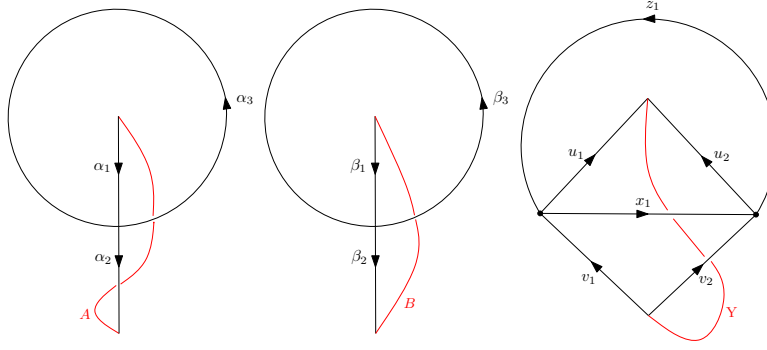


FIGURE 5. Dual graphs of the 3 tube cells

$$Z_{TV}(\mathcal{N}) = \sum \mathcal{D}^{-8} d_{x_1} d_{z_1} d_{\alpha_3} d_{\beta_3} d_{x_2}^{\frac{1}{2}} d_{x_3}^{\frac{1}{2}} d_{z_2}^{\frac{1}{2}} d_{z_3}^{\frac{1}{2}} d_{u_1}^{\frac{1}{2}} d_{u_2}^{\frac{1}{2}} d_{v_1}^{\frac{1}{2}} d_{v_2}^{\frac{1}{2}}$$

Here we have used the pairing of dual graphs and semisimplicity (Lemma 1.3). Using Lemma 1.3 three more times, we get

$$Z_{TV}(\mathcal{N}) = \sum \mathcal{D}^{-8} d_{z_1}^{\frac{1}{2}} d_{x_2}^{\frac{1}{2}} d_{x_3}^{\frac{1}{2}} d_{z_2}^{\frac{1}{2}} d_{z_3}^{\frac{1}{2}} d_{u_1}^{\frac{1}{2}} d_{u_2}^{\frac{1}{2}} d_{v_1}^{\frac{1}{2}} d_{v_2}^{\frac{1}{2}}$$

If we pair some more and cancel opposite twists, this equals

$$\sum \mathcal{D}^{-8} d_{z_1}^{\frac{1}{2}} d_{x_2}^{\frac{1}{2}} d_{x_3}^{\frac{1}{2}} d_{z_2}^{\frac{1}{2}} d_{z_3}^{\frac{1}{2}} d_{u_2}^{\frac{1}{2}} d_{v_1}^{\frac{1}{2}}$$

It follows from Lemma 2.2 and Example 8.6 of [BK] that the loop labeled  $z_1$  projects the top morphism onto  $\text{Hom}_{Z(\mathcal{C})}(1, Y \otimes A \otimes B)$ . After projection, the diagram may be depicted as below where  $P$  is the projector from Lemma 1.7 and the diagram on the left follows from the fact that for a projector,  $(P\varphi, \varphi') = (P\varphi, P\varphi')$ . The final isomorphism may be easily verified by the reader.

$$= (P\varphi', (1_Y \otimes \sigma_{AB})P\varphi)$$

**Proof of Lemma 2.2.** We choose the following combinatorial structure on  $\mathcal{M}$ :

- Three 3-cells, one of which is an open embedded tube, with longitude labeled by  $Y \in \text{Irr}(Z(\mathcal{C}))$ .
- 8 vertices, all of which lie on the boundary
- 15 edges, three of which are internal.

For each 3-cell, we get a graph. Gluing these graphs together using Lemma 1.3

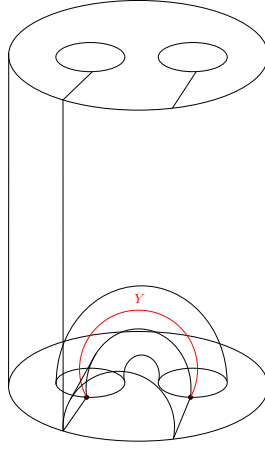
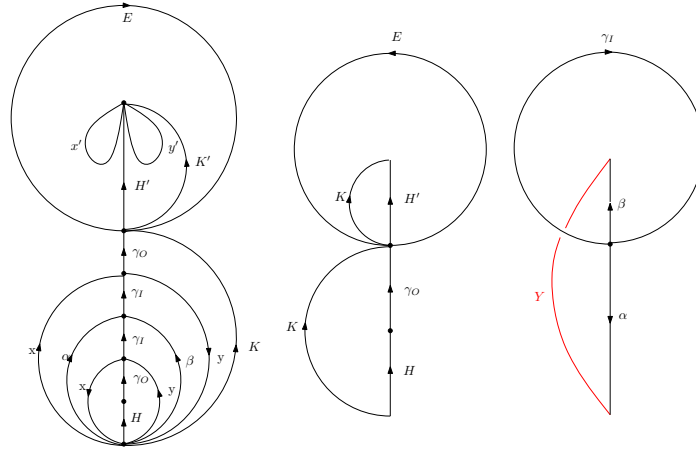
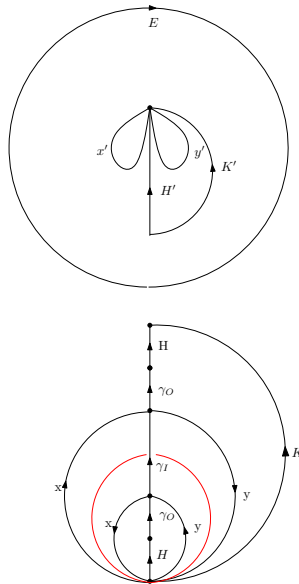
FIGURE 6. Decomposition of  $\mathcal{M}$ 

FIGURE 7. The graphs corresponding to the three 3-cells from Figure 6. From left to right, these correspond to the main 3-cell, the outer 3-cell and the embedded tube cell

yields the following graph:

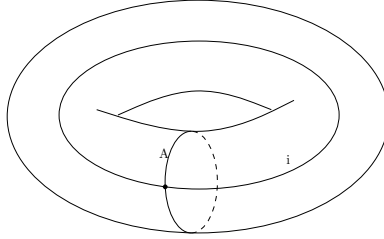


Simplifying the graph using Lemma 1.3, the state-sum formula becomes

$$Z_{TV,C}(M) = \sum \mathcal{D}^{-8} d_E (d_x d_{x'} d_y d_{y'} d_K d_{K'})^{\frac{1}{2}}$$

Using the same arguments as the previous calculation, one can see that this gives  $(P\varphi', ev_Y(P\varphi))$ , where  $ev_Y : Y^* \otimes Y \rightarrow \mathbf{1}$  is the evaluation map in  $Z(\mathcal{C})$ .

**5.1. Computation of  $Z_{TV}(\mathbb{T}^2 \times I)$  from Lemma 3.1.** Let us take the following polytope decomposition of  $\mathbb{T}^2$  consisting of one vertex, two edges and one face. Then



$$H(\mathbb{T}^2) = \bigoplus_{A, X \in Irr \mathcal{C}} \langle A, X, A^*, X^* \rangle.$$

. We will often make use of the following isomorphisms:

$$(5.1) \quad H(\mathbb{T}^2) \cong \bigoplus_{Z, A} \langle Z, A \rangle \otimes \langle Z^*, A^* \rangle \cong \bigoplus_Z \langle Z, Z^* \rangle_{\mathcal{C}}$$

where the first isomorphism is given by the map  $G$  from Lemma 1.4, and the second is given by a direct sum of composition maps (Lemma 1.1). Choosing bases  $\{\varphi_{Z,A,i}\}$  in  $\langle Z, A \rangle$  and  $\{\psi_{Z,A,j}\}$  in  $\langle Z^*, A^* \rangle$ , we can write a basis in  $H(\mathbb{T}^2)$  as

$$\eta_{Z,A,i,j} = \sum_{X \in Irr(\mathcal{C})} \frac{\sqrt{d_X} \sqrt{d_Z}}{\mathcal{D}} \varphi \otimes \psi$$

Composing the map  $G$  with the direct sum of composition maps, we see that  $H(\mathbb{T}^2) \cong \bigoplus_Z \langle Z, Z^* \rangle_{\mathcal{C}}$ . We now show that  $Z_{TV, \mathcal{C}}(\mathbb{T}^2 \times I)$  computes the projection onto  $\bigoplus_{Z \in Irr(Z(\mathcal{C}))} \langle Z, Z^* \rangle_{Z(\mathcal{C})}$ .

Consider the decomposition of  $\mathbb{T}^2 \times I$  shown below. . The top and bottom of the

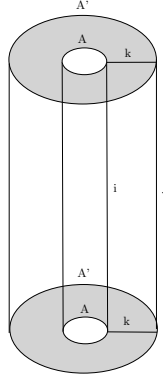
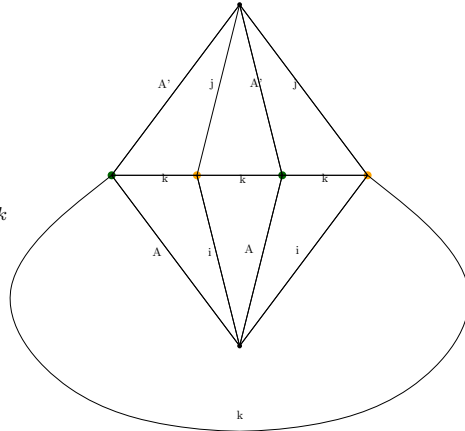


FIGURE 8. Decomposition of  $\mathbb{T}^2 \times I$

figure are shaded to emphasize that they are to be identified. This decomposition consists of 2 vertices both on the boundary, 5 edges 4 of which lie on the boundary, and a single 3-cell.

$$Z_{TV}(\mathbb{T}^2 \times I) = \sum \frac{1}{\mathcal{D}^2} (d_A d_{A'} d_i d_j)^{\frac{1}{2}} d_k$$



where we have labelled vertices dual to one another by the same color. This gives a map  $\Phi : \bigoplus_{A,i} \langle A, X_i, A^*, X_i^* \rangle \longrightarrow \bigoplus_{A',j} \langle A', X_j, A'^*, X_j^* \rangle$ . Composing on both sides



by  $G$  yields a map  $G^{-1}\Phi G : \bigoplus_{Z,A} \langle Z, A \rangle \otimes \langle Z^*, A^* \rangle \longrightarrow \bigoplus_{W,A'} \langle W, A' \rangle \otimes \langle W^*, A'^* \rangle$ .

$$\sum \frac{1}{\mathcal{D}^4} (d_A d_{A'})^{\frac{1}{2}} d_i d_j d_k$$

Pairing the two orange vertices yields

$$\bigoplus_{Z,W} \sum \frac{1}{\mathcal{D}^4} (d_A d_{A'})^{\frac{1}{2}} d_i d_k$$

By Schur's Lemma, this diagram evaluates to 0 unless  $W = Z^*$ . If  $W = Z^*$ , this equals (Lemma 1.7)

$$\bigoplus_Z \sum \frac{1}{\mathcal{D}^2} (d_A d_{A'})^{\frac{1}{2}} d_k = \bigoplus_Z \sum \frac{1}{\mathcal{D}^2} d_k$$

where the equality follows from pairing the dual vertices and using the rescaled composition map twice. By Lemma 1.7 this is a projector onto

$$\bigoplus_{Z \in \text{Irr}(Z(\mathcal{C}))} \text{Hom}_{Z(\mathcal{C})}(\mathbf{1}, Z \otimes Z^*).$$

This computation also shows that the projection is compatible with 5.1.

**Proof of Lemma 4.4.**  $\mathbf{T}_Z^2 \sqcup_U \mathbf{T}_W^2 = S^2 \times S^1$  with two unlinked embedded tubes labelled by  $Z$  and  $W$ . Choose a decomposition of  $S^2 \times S^1$  as pictured in Figure 9. This decomposition has two vertices, five edges and three 3-cells, two of which are embedded tubes. The computation of the state sum is similar to that in Lemma 3.1.

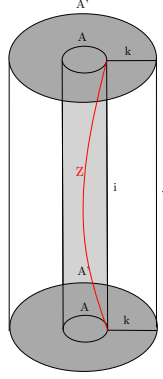


FIGURE 9. Decomposition of  $S^2 \times S^1$  with two closed embedded tubes. The tube labelled  $W$  is to be glued on the outside of the pictured cylinder and the top and bottom of the picture are identified.

We get

$$Z(\mathbf{T}_Z^2 \sqcup_U \mathbf{T}_W^2) = \sum \frac{1}{\mathcal{D}^4} d_A d_{A'} d_i d_k$$

where the orange vertices are labelled by dual vectors. By Lemma 1.7 this equals

$$\delta_{Z,W} \sum_{A,A',k} \frac{1}{\mathcal{D}^2} \frac{d_A d_{A'} d_k}{d_Z}$$

Finally, pairing dual vertices using Lemma 1.3, we get

$$\delta_{Z,W} \sum_{A,A',k} \frac{1}{\mathcal{D}^2} \frac{d_A d_{A'} d_k}{d_Z} = \delta_{Z,W} \sum_{A,k} \frac{1}{\mathcal{D}^2} \frac{d_A d_k}{d_Z} = \delta_{Z,W} \sum_k \frac{1}{\mathcal{D}^2} \frac{d_k}{d_Z} = \delta_{Z,W}$$

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